

Ergodic properties of quantum conservative systems

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Abstract

In this paper we discuss the ergodic properties of quantum conservative systems by analyzing the behavior of two different models. Despite their intrinsic differences they both show localization effects in analogy to the dynamical localization found in Kicked Rotator.

1 Introduction

The study of quantum systems which are classically chaotic is known under the general term of Quantum Chaos. This subject is very interesting and quite new. Indeed, while the statistical properties of chaotic classical systems are well described by the ergodic theory[1], much less is known about Quantum Chaos.

Since the preliminary investigations in the 70's [2] it became quite clear that one of the main characteristics of the classical chaotic motion, namely the diffusion in the phase space, is suppressed by quantum mechanics. Despite the fact that this phenomenon was discovered in a particular time-periodic Hamiltonian system, (the so-called Kicked Rotator model) it was later predicted and observed in other, more physical systems, like the Hydrogen atom in a microwave field[3]. The quantum suppression of classical excitation has been called "dynamical localization"[4].

On the other hand, bounded conservative systems, were analyzed from the point of view of the statistical properties of eigenvalues and eigenvectors[5]. Important analogies were established between the statistical properties of Hamiltonian spectra and those derived from random matrices for which a fairly deep mathematical analysis had been developed[6]. In particular the spacings distribution for nearest neighboring levels of a classically chaotic system was found to be close to that derived from random matrices belonging to a given symmetry class [5] (Gaussian Orthogonal (GOE) or Gaussian Unitary Ensembles (GUE) depending on whether the Hamiltonian is invariant or not under time reversal). Other approaches are based on nonlinear σ -models and supersymmetric techniques.

In this paper we investigate two simple conservative systems in order to show that the dynamical localization can appear in this case also. Roughly speaking, in an ergodic conservative system the motion takes place on the whole energy surface. Correspondingly, in the quantum case, an eigenfunction can be extended or localized inside the energy shell.

A main difficulty in understanding quantum chaotic motion arises from the fact that a bounded conservative quantum system is characterized by a discrete energy spectrum with a quasi-periodic motion and this can be hardly compared with the typical features of the corresponding classical chaotic motion which is characterized by a continuous spectrum. At first glance this would seem a failure of the correspondence principle. However the distinction between discrete and continuous spectrum becomes meaningful only on infinite time. On finite time scales, instead, the quantum motion can be chaotic as the classical one.

A discussion of time scales, which play a fundamental rôle in the study of Quantum Chaos, is given in Section 2. In Section 3 we will analyze a model of classically ergodic conservative system, the so-called Wigner Banded Random Matrix ensemble, where the dynamical localization phenomenon is shown to take place. Finally, in Section 4, we will investigate the problem of localization in a more physical dynamical system : the Bunimovich stadium.

2 Time Scales

What are the physically meaningful time scales in quantum dynamics? To answer the question let us compare the quantum and classical evolution starting from the same initial conditions. Namely, let us consider the evolution of a quantum narrow packet and of a beam of classical orbits initially centered in the same small area of phase space. As it is known, the narrow packet will follow the beam of classical orbits as long as the beam remains narrow. Due to the exponential instability of classical chaotic motion, this can happen only up to a time t_E , called Ehrenfest time,

$$t_E \propto \log(I/\hbar)$$

where I is a typical value of the action of the system. This time scale, first introduced in [7], is very short. Yet, according to the correspondence principle, it grows to infinite as $\hbar \rightarrow 0$. For times $t < t_E$ the quantum dynamics is chaotic as the classical one, including the exponential instability of the motion.

Another, more interesting and longer time scale, is related to the Heisenberg uncertainty principle. Indeed, if ρ^* is the level density of the operative eigenstates involved in the quantum dynamics (namely the eigenstates which enter in the initial condition) then the discreteness of the spectrum can be resolved by quantum dynamics only after a time t_B [4] :

$$t_B \simeq \hbar \rho^* \quad (1)$$

This time scale is related to the process of relaxation to the quantum steady state. Indeed, for times $t < t_B$, quantum and classical averages of dynamical observables will be close to each other while, for $t > t_B$, the quantum system will “see” the discrete nature of the spectrum and consequently it will reach a stationary regime. Quite obviously, as required by the correspondence principle, this time also goes to infinite when $\hbar \rightarrow 0$. Notice, however, that as $\hbar \rightarrow 0$, the time t_B diverges to infinite according to a power law dependence. Therefore the quantum relaxation time t_B is much larger than the time t_E which characterizes the quantum instability of the motion. This means that the quantum diffusion and the relaxation process take place in absence of exponential instability, which is confined in the small time interval t_E .

The nature of the quantum steady state will depend on the comparison between the time t_B and the ergodic time t_{erg} for the classical relaxation to equilibrium. Indeed, if L is a typical length scale over which classical diffusive motion takes place with diffusion rate D , then the ergodic time t_{erg} can be estimated as:

$$t_{erg} \simeq L^2/D \quad (2)$$

It is clear that, if the quantum relaxation time t_B is larger than the classical one t_{erg} , then the quantum steady state will be close (apart quantum fluctuations) to the classical one, given by the microcanonical ensemble. On the other hand, a more interesting situation appears when $t_B < t_{erg}$. In such a case, the quantum distribution will relax to a steady state which is not ergodic but localized around the initial excitation.

In conclusion, the nature of the quantum steady state is determined by the parameter λ , which has been called “ergodicity parameter” :

$$\lambda^2 = t_B/t_{erg} \quad (3)$$

Systems characterized by $\lambda \gg 1$ will relax toward a quantum ergodic steady state, while, for $\lambda \ll 1$, the quantum stationary state is localized. In the following sections we will discuss, on specific examples, the mechanism through which quantum dynamical localization can actually take place in conservative systems.

3 The Wigner Band Random Matrices Ensemble

Apart from few recent papers[8], the main interest in the Random Matrix Theory has been related with full random matrices [6]. However, for real quantum Hamiltonian systems, matrix elements decay on moving away from the main diagonal and this lead to the consideration of matrices with a band structure, in which matrix elements are different from zero only inside a band of size b around the main diagonal.

Let us consider, therefore, the following band random matrix(WBRM) :

$$H_{m,n} = \epsilon_n \delta_{m,n} + v_{m,n} \quad (4)$$

which can be taken to describe an Hamiltonian system of the type $H = H_0 + V$ where H_0 is integrable and V is a perturbation which renders the Hamiltonian H ergodic and mixing. The unperturbed energies ϵ_n are assumed to be distributed according to a Poisson law. The off-diagonal matrix elements $v_{m,n}$ are taken as Gaussian random numbers with zero average and variance v inside a band of size b :

$$\langle v_{m,n}^2 \rangle = v^2 \quad \text{for} \quad |m - n| \leq b. \quad (5)$$

Outside the band b , the matrix elements are zero. This model is then defined by three parameters : b, v and the average density ρ of unperturbed levels:

$$\rho = 1 / \langle \epsilon_n - \epsilon_{n-1} \rangle. \quad (6)$$

This model, introduced by Wigner[9], has nowadays attracted much attention [10, 11], and several interesting results have been obtained. Here, we only briefly mention those connected to the localization problem.

The quantities of interest are the Strength Function or Local Density of States (LDOS):

$$w(E|E_0) = \sum_m \langle |\psi_n(E_m)|^2 \rangle_n \delta(E - E_m) \quad (7)$$

and the averaged eigenfunctions distribution in the energy space:

$$W(E_0|E) = \sum_n \langle |\psi_n(E_m)|^2 \rangle_m \delta(E_0 - E_n^0) \quad (8)$$

Here, $\psi_n(E_m)$ is the n -th component of the eigenfunction having E_m as eigenvalue and the averages ($\langle \dots \rangle_n$, $\langle \dots \rangle_m$) have been done respectively over a small number of n values close to E_0 and over those eigenfunctions having an eigenvalue E_m close to E .

The non-perturbative regime is defined by the condition that the strength of the perturbation must be larger than the average unperturbed levels spacing :

$$\rho v > 1.$$

Above the perturbative regime, the shape of LDOS depends on the Wigner parameter:

$$q = \frac{\rho v^2}{b/\rho}$$

Here the numerator represents the spreading width induced by the perturbation (given by the Fermi golden rule), while the denominator stands for the width, in energy, of the band matrix.

It is possible to show that, when $q \gg 1$, the LDOS has a semicircle form with diameter $\Delta E = 4v\sqrt{2b}$ (semicircle regime (SC)). In the other regime, characterized by $q \ll 1$, the LDOS has a Lorentzian shape with the main part inside a width $\Gamma = 2\pi\rho v^2 \ll \Delta E = 2b/\rho$ (Breit-Wigner regime (BW)).

Due to energy conservation, the number of states occupied by an eigenfunction will be bounded from above by the ergodic localization length $\ell_e \simeq \rho\Delta E$ which gives the maximum number of states which can be coupled by the perturbation. We then consider, as localized, those states with a localization length significantly less than the maximum one ℓ_e , namely we consider localization *inside* the energy shell ΔE . As a measure of the degree of localization we can take, for instance, the inverse participation ratio,

$$\ell = 1 / \sum_n |\psi_n(E_m)|^4 \quad (9)$$

It can be shown[11] that the ratio $\beta_{loc} = \ell/\ell_e$ obeys the following scaling relation:

$$\beta_{loc} = \frac{\ell}{\ell_e} \approx 1 - e^{-\lambda} \quad (10)$$

where the parameter λ , which plays the role here of the “ergodicity parameter”, is defined as

$$\lambda = \frac{\ell_\infty}{\ell_e} \simeq \frac{b^2}{\rho\Delta E} \quad (11)$$

In Eq. (11), ℓ_∞ is the localization length for band random matrices with infinite unperturbed density ρ [12, 13]. The parameter λ is called ergodicity parameter since, when $\lambda \gg 1$, one has $\beta \simeq 1$ and therefore $\ell \simeq \ell_e$. On the other side when $0 < \lambda \ll 1$ one has $\beta \simeq \lambda \ll 1$ and $\ell \ll \ell_e$.

The global properties of the eigenfunctions are also connected with the statistical properties of the spectrum. Indeed, it was found that effects of localization manifest

in the repulsion of neighboring levels. To be more precise, one can show[14] that, in case of localization, the distribution of neighboring levels spacing obeys the following relation:

$$p(s) = As^\beta \exp [- \pi^2 \beta s^2 / 16 - (B - \pi \beta / 4) s] \quad (12)$$

where A, B are constants obtained from normalization. An interesting fact, which so far has not yet received a theoretical explanation, is that the numerical value of the repulsion parameter β in (12) turns out to be very close to the localization parameter β_{loc} [11].

4 The Bunimovich Stadium

In the previous section we have shown that quantum dynamical localization can take place in the model of WBRM. Even though WBRM are believed to describe the qualitative properties of conservative, classically chaotic, Hamiltonian systems, it is highly desirable to analyze a more realistic Hamiltonian system. To this end, two-dimensional billiards are very convenient objects to study since they have very clean mathematical properties, from complete integrability (e.g. the circle) to complete chaotic motion (e.g. a dispersive billiard). Moreover their classical and quantum dynamics can be numerically studied with sufficient good accuracy. Finally, modern laboratory techniques allow for quite accurate experimental investigations.

In the following we consider a well known billiard model, the so-called Bunimovich Stadium, which consists of two semicircles, with radius $R = 1$, connected by two straight lines with length $2a$. Inside this bounded two-dimensional region we consider the motion of a point particle with mass $m = 1$ colliding elastically with the boundary. The classical dynamics depends only on the ratio $\epsilon = a/R$ and it can be rigorously proven to be ergodic and mixing for any $\epsilon \neq 0$. When $\epsilon \sim 1$ the relaxation time to statistical equilibrium is very short, just few collisions with the boundary. Here we are interested to the case $\epsilon \ll 1$ when the stadium is very close to the circle. For the billiard in a circle, there exist two constants of motion, the energy $E = m\vec{v}^2/2$ and the angular momentum $\vec{l} = l_z \hat{k}$ (here \hat{k} is the unit vector perpendicular to the plane of the stadium and identifying the z axis). For $\epsilon > 0$ the rotational symmetry around the z -axis is broken and l_z is not a constant of motion any more. Nevertheless, if ϵ is sufficiently small, one can expect that the angular momentum will change slowly in time. Indeed, the angular momentum can vary in the interval $|l_z| < l_{max} = m(R + a)|\vec{v}| \simeq \sqrt{2mR^2E}$; now, if for example at $t = 0$ the angular momentum is zero, then it will evolve in time in a diffusive way, with diffusion coefficient D , until the system will reach the equilibrium state given by the microcanonical ensemble. The ergodic time t_{erg} , namely the time necessary to reach the statistical equilibrium, can be estimated as:

$$t_{erg} \simeq l_{max}^2 / D \quad (13)$$

The diffusion coefficient D can be computed numerically [15]. In terms of the rescaled angular momentum $L = l_z/l_{max}$, the diffusion coefficient, in the number of collisions with the boundary, is given by $D_0 = 1.5 \epsilon^{5/2}$. By taking into account that the time interval between two successive collisions is $t_c \sim E^{-1/2}$ one has that, neglecting numerical constants ($m = R = 1$),

$$t_{erg} \sim \frac{E}{E\epsilon^{5/2}} t_c \sim \epsilon^{-5/2} E^{-1/2} \quad (14)$$

The classical dynamics of the billiard can be approximated by the following area-preserving map which gives the change of L and of its related conjugated variable θ , between two successive collisions with the boundary[15]:

$$\begin{aligned} \bar{L} &= L + \varepsilon \sin \theta \operatorname{sgn}(\cos \theta) \\ \bar{\theta} &= \theta + \pi - 2 \arcsin \bar{L} \end{aligned} \quad (15)$$

Here $\varepsilon = -2\epsilon \operatorname{sgn}(L_0)\sqrt{1 - L_0^2}$ and L_0 is the initial value of the angular momentum.

The map (15) represents a first order approximation to the real dynamics when the initial rescaled angular momentum L_0 is not too large. It also give rise, for $\varepsilon \ll 1$, to a diffusive motion with a diffusion rate $D \propto \varepsilon^{5/2}$ in agreement with numerical computations on the real billiard.

The map description allows to understand the properties of the quantum dynamics in analogy to the Kicked Rotator model[2]. Indeed we know that, above the quantum perturbative regime $\varepsilon > \hbar$, quantum dynamics will follow the classical diffusive motion up to a certain time, called break-time, (see also Eq.(1)) given by $\tau_b \sim D_0[4]$, where τ_b is measured in the number of collisions and D_0 is the dimensionless diffusion coefficient. In physical units, this time is given by:

$$t_B \sim \epsilon^{5/2} E^{1/2} / \hbar^2 \quad (16)$$

This allows to estimate the ergodicity parameter λ (Eq.(3))

$$\lambda^2 = t_B / t_{erg} = E\epsilon^5 / \hbar^2 \quad (17)$$

The value $\lambda = 1$ gives the critical energy value:

$$E_{erg} \sim \hbar^2 / \epsilon^5 \quad (18)$$

above which we expect quantum ergodic behavior. On the contrary, for energies $E < E_{erg}$, we expect dynamical localization. Since the total average number of states up to the energy E [5] is given by

$$\langle \mathcal{N}(E) \rangle \approx \frac{m\mathcal{A}}{2\pi\hbar^2} E \quad (19)$$

where \mathcal{A} is the area of the billiard, one can estimate the level number $N_{erg} \sim \epsilon^{-5}$ above which quantum ergodic behavior is expected. Numerical computations show that, in agreement with the above estimates, only above N_{erg} the levels spacing distribution is very close to a Wigner–Dyson distribution, characteristic of the GOE ensemble.

One can also numerically compute the repulsion parameter β , using Eq.(12) and one finds [16]

$$\beta \simeq 1 - e^{-\lambda} \quad (20)$$

where λ is the ergodicity parameter (17). This quite surprising result reinforces the similarity between WBRM and the quantum behavior of classically chaotic systems.

Finally, we would like to notice that the ergodicity parameter λ turns out to be proportional to the dimensionless conductance:

$$g = \frac{E_c}{\Delta} \quad (21)$$

where E_c is the Thouless energy and Δ is the average levels spacing. Indeed taking into account that

$$E_c \simeq \frac{\hbar}{t_{erg}} \quad (22)$$

and

$$\Delta \simeq \frac{\hbar}{t_H} \quad (23)$$

where $t_H \simeq \hbar \frac{\mathcal{N}(E)}{E} \simeq \hbar^{-1}$ (see Eq.(19)) is the Heisenberg time; one gets

$$g = \frac{t_H}{t_{erg}} \simeq \frac{1/\hbar}{\epsilon^{-5/2} E^{-1/2}} \propto \lambda. \quad (24)$$

5 Conclusions

In this paper we have discussed the problem of localization in conservative, classically chaotic, Hamiltonian systems. The existence of localization in such systems would restrict quantum distributions to smaller regions of phase space than classically allowed, and would therefore introduce significant deviations from ergodicity. As surmised in [17], this lack of quantum ergodicity may lead to interesting consequences for quantum equilibrium statistical distributions.

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